Fractional Hedonic Games: Individual and Group Stability

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ABSTRACT

Coalition formation provides a versatile framework for analyzing cooperative behavior in multi-agent systems. In particular, hedonic coalition formation has gained considerable attention in the literature. An interesting class of hedonic games recently introduced by Aziz et al. [3] are *fractional hedonic games*. In these games, the utility an agent assigns to a coalition is his average valuation for the members of his coalition. Three common notions of stability in hedonic games are core stability, Nash stability, and individual stability. For each of these notions we show that stable partitions may fail to exist in fractional hedonic games. For core stable partitions this holds even when all players only have symmetric zero/one valuations ("mutual friendship"). We then leverage these counter-examples to show that deciding the existence of stable partitions (and therefore also computing stable partitions) is NP-hard for all considered stability notions. Moreover, we show that checking whether the valuation functions of a fractional hedonic game induce strict preferences over coalitions is coNP-complete.

Categories and Subject Descriptors

[**Theory of computation**]: Algorithmic game theory; [**Theory of computation**]: Solution concepts in game theory; [**Theory of computation**]: Computational complexity and cryptography; [**Computing methodologies**]: Multi-agent systems; [**Mathematics of computing**]: Graph theory

General Terms

Economics, Theory, and Algorithms

Keywords

Cooperative games; coalition formation; hedonic games; computational complexity

1. INTRODUCTION

Hedonic games—as introduced by Drèze and Greenberg [12] and further explored by many others [e.g., 4, 9, 6, 11, 13, 14, 7, 1, 2]—present a natural versatile framework to study the formal aspects of coalition formation. In hedonic games, coalition formation is approached from a game-theoretic angle. The outcomes are coalition structures—partitions of the agents—over which the agents have preferences. Moreover, the agents have different individual or joint strategies at their disposal to affect the coalition structure to be formed. Various solution concepts—such as the *core*, the *strict core*, and several kinds of *individual stability*—have been proposed to analyze these games.

The characteristic feature of hedonic games is a non-externalities condition, which incorporates the useful but arguably simplifying assumption that every agent's preferences over the coalition structures are fully determined by his preferences over coalitions he belongs to and do not depend on how the remaining agents are grouped. Nevertheless, the number of coalitions an agent can be a member of is exponential in the total number of agents and the development and analysis of concise representations as well as interesting subclasses of hedonic games are an ongoing concern in computer science and game theory. Particularly prominent in this respect are representations in which the agents are assumed to entertain preferences over the other agents, which are then systematically lifted to preferences over coalitions [see e.g., 9, 1].

The work presented in this paper pertains to *fractional hedonic games (FHGs)*, a subclass of hedonic games in which every agent is assumed to have cardinal utilities or valuations for the other agents. These induce preferences over coalitions by considering the *average valuation* for the members of every coalition. The higher this valuation, the more preferred the respective coalition is. In other work, the min, max, and sum operators have been used for *hedonic games based on worst agents* [9], *hedonic games based on best agents* [8], and *additively separable hedonic games*, respectively [see, e.g., 2].

FHGs can be represented by a weighted directed graph where the weight of edge (i, j) denotes the valuation of agent i for agent j. The formal study of FHGs was initiated by Aziz et al. [3] who obtained results for core stability in various subclasses of FHGs.

Some natural economic scenarios can be adequately modeled as FHGs. Consider, for example, the formation of political parties. The valuation of two agents for each other may be interpreted as to which extent their opinions overlap, e.g., the inverse of their distance in the political spectrum. In political environments, agents need to form coalitions and join parties to acquire influence. On the other hand, as parties become larger, the disagreement among their members rises,

Appears in: *Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2015), Bordini, Elkind, Weiss, Yolum (eds.), May 4–8, 2015, Istanbul, Turkey.*

	unrestricted	symmetric	simple symmetric
IS	− (NP-c.)	?	+
NS	− (NP-c.)	− (NP-c.)	+
CS	− (NP-h.)	− (NP-h.)	−

Table 1: Summary of results. "+" indicates that the existence of stable partitions is guaranteed for the respective class of games, "−" indicates that there are FHGs in the respective class of games in which no stable partition exists, and "NP-h." and "NP-c." indicate NP-hardness and NP-completeness of deciding whether a stable partition exists, respectively. Aziz et al. [3] showed that core stable partitions in unrestricted FHGs may *not* exist and Bilò et al. [5] showed that Nash stable partitions in simple symmetric FHGs *always* exist.

making them susceptible to split-offs. Thus, one could assume that agents seek to maximize the *average* agreement with the members of their coalition.

In this paper, we study stable partitions in three hierarchical nested subclasses of FHGs: *unrestricted* FHGs (arbitrary valuations), *symmetric* games (mutually equal valuations), and *simple* symmetric games (zero/one valuations). Simple games, as considered by Aziz et al. [3], can be conveniently represented as directed graphs.

Our contribution is twofold. First we study for various stability notions whether a stable partition always exists. We consider core stability, Nash stability, and individual stability. The latter two are based on movements of a single agent, whereas core stability allows a group of agents to deviate. We provide a clear picture for which stability notions a stable partition may fail to exist in the three subclasses of FHGs introduced above.

In the second part of the paper we examine the computational complexity of deciding whether a stable partition exists in a given FHG. Our results suggest a strong connection to the existence results obtained in the first part. More precisely, we could show for several cases that when a stable partition may fail to exist for some stability notion in some class of FHGs, it is NP-hard to decide whether a given game in this class admits a stable partition. This also implies hardness of the important problem of computing a stable partition and stands in sharp contrast to several subclasses of FHGs considered by Aziz et al. [3], where existence of a stable partition was always associated with an efficient algorithm for computing it. We also show that checking whether the valuation functions of an FHG induce strict preferences over coalitions is coNP-complete. Our main findings are summarized in Table 1.

2. RELATED WORK

FHGs were studied for the first time by Aziz et al. [3], who focused on core stability. They show that some FHGs fail to admit a core stable partition and that for various subclasses of FHGs, e.g., games given by complete multipartite graphs or games given by undirected trees, a core stable partition always exists. Aziz et al. left open whether *simple symmetric* FHGs always admit a core stable partition. We answer this question in the negative. Bilò et al. [5] started to analyze FHGs from the viewpoint of non-cooperative game theory. They show that Nash stable partitions may not exist. Furthermore, they give bounds on the price of anarchy and the price of stability. For FHGs given by unweighted graphs the grand coalition is always Nash stable, hence, they examine whenever finer Nash stable partitions exist in these games.

Our work is connected to both papers. We advance the results for core stability and Nash stability and initiate the study of individual stability—a weakening of Nash stability. In particular, we show that, even for very restricted subclasses of FHGs, core stable and Nash stable partitions may not exist. For games in these classes it turns out to be NP-hard to decide whether a stable partition exists.

FHGs are related to *additively separable* hedonic games [see e.g., 2]. In both, FHGs and additively separable hedonic games, every agent has a cardinal valuation for every other agent. In additively separable hedonic games, the valuation for a coalition is derived by adding the valuations for all agents in the coalition. By contrast, in FHGs, the valuation for a coalition is derived by adding the valuations for all agents in the coalition and then dividing the sum by the total number of agents in the coalition. Although conceptually additively separable hedonic games and FHGs are similar, their formal properties are quite different. For example, in unweighted and undirected graphs, the grand coalition is trivially core stable for additively separable hedonic games, which is not the case for FHGs. An FHG approach to social networks with only non-negative weights may therefore help to detect like-minded and densely connected communities. Aziz et al. [3] discuss the relationship between FHGs and network clustering in more detail.

Stability in hedonic games gives rise to many computationally interesting problems, e.g., deciding the existence of, verifying the stability of, and finding stable partitions. These questions were extensively studied in the context of core stability [see, e.g., 20, 22] and additively separable hedonic games [see, e.g., 2, 21]. Aziz et al. [3] showed hardness of two decision problems for core stability in FHGs.

Recently, Olsen [18] has examined a variant of simple symmetric FHGs, i.e., games represented by an unweighted, undirected graph, and investigated the computation and existence of Nash stable outcomes. In the games he considered, however, every maximal matching is core stable and every perfect matching is a best possible outcome even if there are large cliques present in the graph. By contrast, in our setting agents have an incentive to form large cliques.

FHGs are different from but related to another class of hedonic games called *social distance games* introduced by Branzei and Larson [7]. In social distance games, an agent not only derives utility from agents he likes directly but also from agents which are at smaller distances from him.

FHGs also exhibit some similarity with the segregation and status-seeking models considered by Milchtaich and Winter [17] and Lazarova and Dimitrov [16]. Group formation models based on types were first considered by Schelling [19].

3. PRELIMINARIES

Let N be a set $\{1, \dots, n\}$ of *agents* or *players*. A *coalition* is a subset of the agents. For every agent $i \in N$, we let $\mathcal{N}_i$ denote the set $\{S \subseteq N : i \in S\}$ of coalitions i is a member of. Every agent i is equipped with a reflexive, complete, and transitive *preference relation* $\succsim_i$ over the set $\mathcal{N}_i$. We use $\succ_i$ and $\sim_i$ to refer to the strict and indifferent parts of $\succsim_i$, respectively. If $\succsim_i$ is also anti-symmetric we say that i's preferences are *strict*. A coalition $S \in \mathcal{N}_i$ is *acceptable* for an agent i if i weakly prefers S to being alone, i.e., $S \succsim_i \{i\}$ and *unacceptable* otherwise. A *hedonic game* is a pair $(N, \succsim)$, where $\succsim = (\succsim_1, \ldots, \succsim_n)$ is a profile of preference relations $\succsim_i$, modeling the preferences of the agents.

The *valuation function* of an agent i is a function $v_i \colon N \to \mathbb{R}$ assigning a real value to every agent. A valuation function v_i can be extended to a valuation function over coalitions where, for all $S \in \mathcal{N}_i$,

$$v_i(S) = \frac{\sum_{j\in S} v_i(j)}{|S|}.$$

A hedonic game $(N, \succsim)$ is said to be a *fractional hedonic game (FHG)* if, for every agent i in N, there is a valuation function v_i such that for all coalitions $S, T \in \mathcal{N}_i$,

$$S \succsim_i T \text{ if and only if } v_i(S) \geq v_i(T).$$

Hence, every FHG can be compactly represented by a tuple of valuation functions $v = (v_1, \ldots, v_n)$. It can be shown that every FHG can be induced by valuation functions with $v_i(i) = 0$ for all $i \in N$. Thus, we assume $v_i(i) = 0$ throughout the paper. We will frequently associate FHGs with weighted digraphs $G = (N, N \times N, v)$ where the weight of the edge (i, j) is $v_i(j)$, i.e., the valuation of agent i for agent j.

Besides from unrestricted FHGs, two classes of FHGs will be of particular interest to us.

- An FHG is *symmetric* if $v_i(j) = v_j(i)$ for all $i, j \in N$.
- An FHG is *simple* if $v_i(j) \in \{0, 1\}$ for all $i, j \in N$.

We note that FHGs are not a subclass of additively separable hedonic games nor *vice versa*, i.e., there are FHGs that are not additively separable and *vice versa*.

The outcomes of hedonic games are *partitions* of the agents, or coalition structures. Given a partition $\pi = \{\pi_1, \ldots, \pi_m\}$ of the agents, $\pi(i)$ denotes the coalition in π of which agent i is a member. We also write $v_i(\pi)$ for $v_i(\pi(i))$, which reflects the hedonic nature of the games we consider. By the same token we obtain preferences over partitions from preferences over coalitions. We refer to $\{N\}$ as the grand coalition.

Hedonic games are analyzed using solution concepts, which formalize desirable or optimal ways in which the agents can be partitioned (as based on the agents' preferences over the coalitions). In this paper, we consider three notions of stability.

- We say that a coalition $S \subseteq N$ *blocks* a partition π, if every agent $i \in S$ strictly prefers S to his current coalition $\pi(i)$, i.e., if $S \succ_i \pi(i)$ for all $i \in S$. A partition that is not blocked by any coalition is *core stable (CS)*.
- A partition π is *Nash stable (NS)* if no agent can benefit from joining another (possibly empty) coalition, i.e., if $\pi(i) \succsim_i S \cup \{i\}$ for all $S \in \pi \cup \{\emptyset\}$ and $i \in N$.
- A partition π is *individually stable (IS)* if no agent can benefit from joining another (possibly empty) coalition without making some member of the coalition he joins worse off, i.e., if $\pi(i) \succsim_i S \cup \{i\}$ or $S \succ_j S \cup \{i\}$ for some $j \in S$ for all $S \in \pi \cup \{\emptyset\}$ and $i \in N$.

Note that no partition where one agent is placed in an unacceptable coalition is core stable, Nash stable, or individually stable, since this agent could benefit from forming his own coalition, i.e., join the empty coalition.

By definition, every Nash stable partition is also individually stable. However, there is no logical relationship between core stability and any of the remaining stability notions defined above. In particular, there exist core stable partitions which are not individually stable and Nash stable partitions which are not core stable. Bogomolnaia and Jackson [6] provided an example for the first statement and the second statement can be deduced from Example 1.

Example 1. Consider the symmetric FHG given in Figure 1. Agents are represented by vertices and the valuations function by weighted edges. The only core stable partition is $\{\{1, 4\}, \{2, 3\}\}$. This partition is also individually stable but not Nash stable, since agent 4 can benefit from joining the coalition $\{2, 3\}$. On the other hand, the partition $\{\{1\}, \{2, 3, 4\}\}$ is Nash (and hence individually) stable, but not core stable, since it is blocked by the coalition $\{2, 3\}$. If this game where to be interpreted as an additively separable hedonic game, the grand coalition would be core stable and Nash stable.

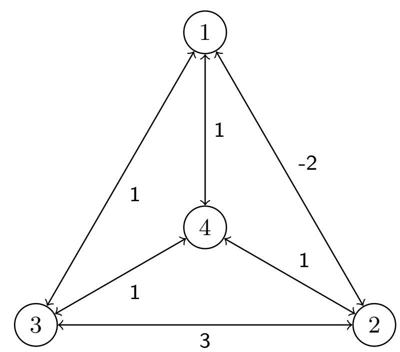

Figure 1: Example of a symmetric FHG.

4. EXISTENCE OF STABLE PARTITIONS

This section is divided into three subsections, each corresponding to one of the classes of FHGs defined above. In these sections we discuss the existence of core stable, Nash stable, and individually stable partitions, respectively.

4.1 Unrestricted FHGs

Aziz et al. [3] and Bilò et al. [5] showed that core stable partitions and Nash stable partitions may not exist in unrestricted FHGs. Our first result is that individually stable partitions also may not exist in unrestricted FHGs.

THEOREM 1. *In unrestricted FHGs, core stable, Nash stable, or individually stable partitions may not exist.*

PROOF. The FHG given in Figure 2 was used by Aziz et al. [3] to show that core stable partitions may not exist in unrestricted FHGs. Furthermore it does not admit a Nash stable or individually stable partition. We show that no individually stable partition exists. This directly implies that no Nash stable partition exists. Note that no partition containing a coalition with three or more agents is individually stable, since it is unacceptable for all its members.

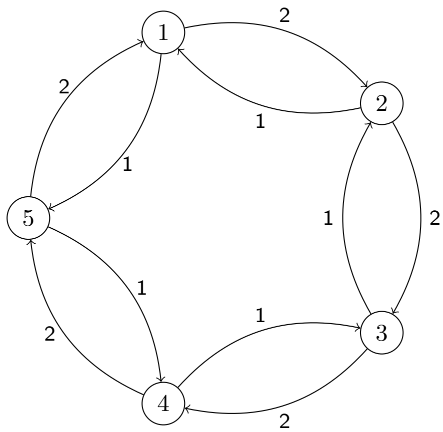

Figure 2: An FHG in which no core stable, Nash stable, or individually stable partition exists. All missing edges have weight -4.

Also, no partition in which two agents i and $i+1$ are in a singleton coalition is individually stable, since i would join $i+1$ and $i+1$ would permit it (or *vice versa*). Hence, up to symmetries, the partition $\pi_1 = \{\{1,2\},\{3,4\},\{5\}\}$ is the only remaining candidate for an individually stable partition. But π_1 is not individually stable, since agent 4 can benefit from joining the singleton coalition $\{5\}$ and agent 5 would permit it. $\square$

4.2 Symmetric FHGs

Symmetry captures the idea that agents mutually benefit from each other to the same extent. Many economic scenarios that can be adequately modeled as FHGs naturally exhibit symmetry. In our introductory example, the valuation of two agents for each other is determined by their distance in the political spectrum. Since distance functions are symmetric by definition, the associated FHG is symmetric, too. We show that even in the context of symmetric FHGs, both core stable partitions and Nash stable partitions may not exist.

THEOREM 2. *In symmetric FHGs, core stable or Nash stable partitions may not exist.*

PROOF. For both statements, we provide games in which no stable partition exists. In the FHG depicted in Figure 3 no core stable partition exists. The proof is omitted, since we prove a stronger statement in Theorem 3.

In the FHG depicted in Figure 4 no Nash stable partition exists. First, note that no partition with agents 2 and 3 in the same coalition is stable, since it is unacceptable for both 2 and 3. Furthermore, no partition in which agent 1 is in a singleton coalition is stable, since he prefers every coalition to being alone. Hence, up to symmetries, we only have to consider $\pi_1 = \{\{1,2\},\{3,4\}\}$, $\pi_2 = \{\{1,4\},\{2\},\{3\}\}$, $\pi_3 = \{\{1,2,4\},\{3\}\}$, and $\pi_4 = \{\{1,2\},\{3\},\{4\}\}$. π_1 is not stable because agent 1 can benefit from joining $\{3,4\}$, π_2 is not stable because agent 2 may join $\{1,4\}$, π_3 is not stable because agent 4 may join $\{3\}$, and π_4 is not stable because agent 3 may join $\{4\}$. $\square$

This result is in contrast to the corresponding statement for additively separable hedonic games. Bogomolnaia and

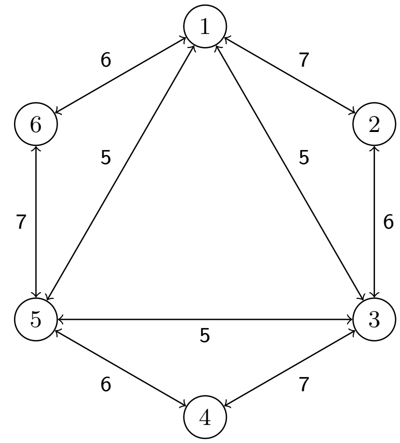

Figure 3: A symmetric FHG in which no core stable partition exists. All missing edges have weight -24.

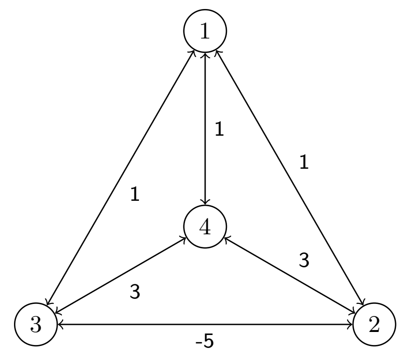

Figure 4: A symmetric FHG in which no Nash stable partition exists.

Jackson [6] proved that every symmetric additively separable hedonic game admits a core stable partition. It remains open whether every symmetric FHG admits an individually stable partition.

4.3 Simple FHGs

In many applications it is reasonable to assume that the agents' valuations only take the values zero and one. This is, for example, the case in social networks or exchange economies, if agents only distinguish between non-friends and friends. These so-called simple games can be represented as *unweighted* directed graphs. There is an edge from one agent to another if the former has valuation one for the latter. Aziz et al. [3] considered bakers and millers games, which form a subclass of simple games. These games correspond to *complete* 2-partite graphs. More generally, they show that for games that correspond to complete k-partite graphs a core stable partition always exists. An open question proposed by Aziz et al. [3] is whether every simple *symmetric* FHG admits a core stable partition. We answer this question in the negative by providing a counter-example with 40 agents.[1]

[1]When only requiring non-negative and symmetric valuations, there is a counter-example with only 15 agents.

Theorem 3. *In simple symmetric FHGs, core stable partitions may not exist.*

Proof. The FHG depicted in Figure 5 does not admit a core stable partition. For two agents $i, j \in N$ we say that i is connected to j if i's valuation for j is 1 (and *vice versa*). Let π be a core stable partition. The first step is to show that $A_l \subseteq S \in \pi$ and $C_l \subseteq T \in \pi$ for all $l \in \{1, \ldots, 5\}$. We show both statements for $l = 1$. The rest follows from the symmetry of the game.

$A_1 \subseteq S \in \pi$: Assume for contradiction that this is not the case. Since $A_1 \cup C_1$ is a 6-clique, at least one agent $i \in A_1 \cup C_1$ has a valuations of at least $5/6$ for his coalition (otherwise $A_1 \cup C_1$ is blocking). Assume $i \in A_1$. If $\pi(i)$ contains an agent i is not connected to, then $u_i(\pi) \leq 9/11 < 5/6$ since i is connected to at most 9 agents in any coalition. Hence $\pi(i)$ only contains agents i is connected to. But then $A_1 \cup \pi(i)$ is blocking, since every agent in A_1 is connected to the same agents as i, a contradiction. Hence $i \in C_1$. $A_1 \cap \pi(i) = \emptyset$ implies $u_i(\pi) \leq 4/5$. If $\pi(i)$ contains an agent i is not connected to, then $u_i(\pi) \leq 7/9 < 5/6$, since i is connected to at most 7 agents in any coalition. Hence, $\pi(i) \cap A_1 = S \neq \emptyset$ and $\pi(i)$ only contains agents i is connected to. Thus, $C_1 \subseteq \pi(i)$ (otherwise $C_1 \cup \pi(i)$ is blocking).

At least one agent k_1 in $A_1 \cup B_1$ and at least one agent k_2 in $A_1 \cup B_5$ has a valuation of at least $4/5$ for his coalition, since both sets are 5-cliques. $k_1, k_2 \notin S$, since $u_j(\pi) \leq 4/6$ for all $j \in S$. If $k_1 \in A_1 \setminus S$, then $\pi(k_1)$ only contains agents k_1 is connected to, otherwise $u_{k_1}(\pi) \leq 5/7 < 4/5$. Then $\pi(k_1) \cup S$ is blocking. Hence $k_1 \in B_1$. Analogously it follows that $k_2 \in B_5$. We show that $\pi(k_1) \neq \pi(k_2)$. Assume for contradiction that $\pi(k_1) = \pi(k_2) = T$. If S contains at least two agents k_1 is not connected to, we have $u_{k_1}(T) \leq 10/13 < 4/5$ (since k_1 is connected to at most 10 agents in any coalition). Hence, T contains one agent k_1 is not connected to, namely k_2. The analogous assertion holds for k_2. Since $u_{k_2}(T) \geq 4/5$, we have $|T| \geq 10$. But then T contains at least 2 agents k_1 is not connected to, since there are only 3 agents that both k_1 and k_2 are connected to. This implies that $u_{k_1} \leq 10/13 < 4/5$, a contradiction.

If $B_4 \subseteq \pi(i)$, it follows that $u_j(\pi) \leq 4/7 < 2/3$ for all $j \in S$. If $j \in A_1 \setminus S$ is in a coalition with an agent j is not connected to, $u_j(\pi) \leq 3/5 < 2/3$, since $\pi(j)$ cannot contain an agent from $\pi(i)$ and from both $\pi(k_1)$ and $\pi(k_2)$ (since $\pi(k_1) \neq \pi(k_2)$). Hence $S \cup \pi(j)$ is blocking.

If $|\pi(i) \cap B_4| = 1$ it follows that $|S| = 2$ and $u_j(\pi) = 4/6$ for all $j \in S$. At least one agent k in $A_5 \cup B_4$ has a valuation of at least $4/5$ for his coalition. If $k \in \pi(i)$ it follows that $u_k(\pi) = 3/6 < 4/5$, a contradiction. If $k \in B_4 \setminus \pi(i)$, then $\pi(k)$ only contains agents k is connected to. If $A_4 \subseteq \pi(k)$ or $A_5 \subseteq \pi(k)$, then $A_4 \cup C_4$ $A_5 \cup C_5$ are blocking, respectively. $|\pi(k) \cap A_4| = 2$ or $|\pi(k) \cap A_5| = 2$ is not possible since our previous analysis for A_1 and C_1 also applies to A_4 and C_4, and A_5 and C_5, respectively. But then, $u_k(\pi) \leq 2/3 < 4/5$. Hence $k \in A_5$. This implies that $\pi(k)$ only contains agents k is connected to. Hence $A_5 \subseteq \pi(k)$, otherwise $A_5 \cup \pi(k)$ is blocking. Also $\pi(k) \neq A_5 \cup B_5$, because otherwise $A_5 \cup C_5$ is blocking. Hence $A_5 \cap \pi(k_2) = \emptyset$. Thus, $\pi(k_2)$ can only contain agents k_2 is connected to. This implies $B_5 \subseteq \pi(k_2)$. As $u_{k_2}(\pi) \geq 4/5$, $|\pi(k_2) \cap C_2| \geq 2$. Thus, if $\pi(k_1)$ contains some agent in A_2, then $A_2 \cup C_2$ is blocking. If $\pi(k_2) = B_5 \cup C_2$, then $A_2 \cup C_2$ is blocking and otherwise C_2 is blocking. This contradicts that π is stable.

$C_1 \subseteq T \in \pi$: At least one agent i in $B_4 \cup C_1$ has a valuation of at least $4/5$ for his coalition (otherwise $B_4 \cup C_1$ is blocking). Assume $i \in B_4$ and $u_j(\pi) < 4/5$ for all $j \in C_1$. Then $A_l \subseteq \pi(i)$ for some $l \in \{4, 5\}$. But then $A_l \cup C_l$ is blocking. Hence the assumption is wrong and $i \in C_1$. Note that $\pi(i)$ cannot contain an agent i is not connected to, otherwise $u_i(\pi) \leq 7/9 < 4/5$, since i is connected to at most 7 agents in any coalition. But then $C_1 \subseteq \pi(i)$, otherwise $C_1 \cup \pi(i)$ is blocking.

It cannot be that $\pi(i) \subseteq A_l \cup B_l$ or $\pi(i) \subseteq A_l \cup B_{l-1}$ for $i \in A_l$, since $A_l \cup C_l$ is blocking for all $l \in \{1, \ldots, 5\}$.

If $A_1 \cup C_1 \cup S \in \pi$ with $\emptyset \neq S \subseteq B_4$, then $u_i(\pi) \leq 5/7 < 4/5$ for all $i \in A_1$. Hence $B_1 \cup C_3, B_5 \cup C_2 \in \pi$, otherwise either $A_1 \cup B_1$ or $A_1 \cup B_5$ are blocking ($u_i(A_1 \cup B_1) = u_i(A_1 \cup B_5) = 4/5$ for all $i \in A_1$). But then $u_i(\pi) \leq 4/5$ for all $i \in A_2$. Hence $A_2 \cup C_2$ is blocking, a contradiction. In any other partition in which some $i \in A_1$ is in a coalition with an agent he is not connected to, we have $u_i(\pi) \leq 9/11 < 5/6$ for all $i \in A_1$ and $u_j(\pi) \leq 4/5 < 5/6$ for all $j \in C_1$. Hence $A_1 \cup C_1$ is blocking. Hence $\pi(i)$ only contains agents i is connected to for all $i \in A_l$ and $l \in \{1, \ldots, 5\}$.

We have shown previously that at least one agent $i_l \in C_l$ has a valuation of at least $4/5$ for his coalition for all $l \in \{1, \ldots, 5\}$. Hence, $\pi(i_l)$ cannot contain an agent i_l is not connected to. Therefore, either $\pi(i_l) = A_l \cup C_l$ or $\pi(i_l) = B_{l-2} \cup C_l$ for all $l \in \{1, \ldots, 5\}$. If $A_l \cup C_l \in \pi$ for all $j \in \{1, \ldots, 5\}$, then $A_1 \cup B_1 \cup B_5$ is blocking. Hence we can assume without loss of generality that $A_1 \cup S \in \pi$ with $S \subseteq B_1 \cup B_5$. If $|S| < 3$, then $A_1 \cup C_1$ is blocking. Hence $|S| \geq 3$. Without loss of generality, $B_5 \subseteq S$. If follows that $B_4 \cup C_1 \in \pi$, since one agent in C_1 has a valuation of at least $4/5$ for his coalition. This implies that $A_4 \cup C_4 \in \pi$. Furthermore $A_2 \cup C_2, A_3 \cup C_3 \in \pi$, otherwise $B_5 \cup C_2$ or $B_1 \cup C_3$ are blocking. Then we get $A_5 \cup C_5 \in \pi$. But then $A_3 \cup B_2 \cup B_3$ is blocking. Hence, π is not core stable, a contradiction. □

5. COMPUTATIONAL COMPLEXITY

We now focus on the computational complexity of various decision problems associated with FHGs. Since the number of coalitions an agent can be a member of is exponential in the number of agents, we assume in this section that the agents' preferences are not given explicitly as rankings over coalitions but implicitly by valuation functions. First, we will show that, given the valuation functions, it is coNP-complete to decide whether every agent has strict preferences over coalitions.

Loosely put, our second main result in this section shows that whenever in some class of games a stable partition may fail to exist for some stability notion, it is NP-hard to decide whether a stable partition exists for a game in this class. The proof is based on a generic construction which uses the counter-examples from Section 4 as gadgets.

5.1 Strictness of Preferences

A problem of independent interest is to decide for given valuation functions whether these induce strict preferences over coalitions. Clearly, it is easy to verify that an agent is indifferent between two coalitions. But since the number of coalitions is exponential in the number of agents, it is not clear how to efficiently verify that no agent is indifferent between some pair of coalitions. We will show that it is unlikely that an efficient algorithm for this problem exists.

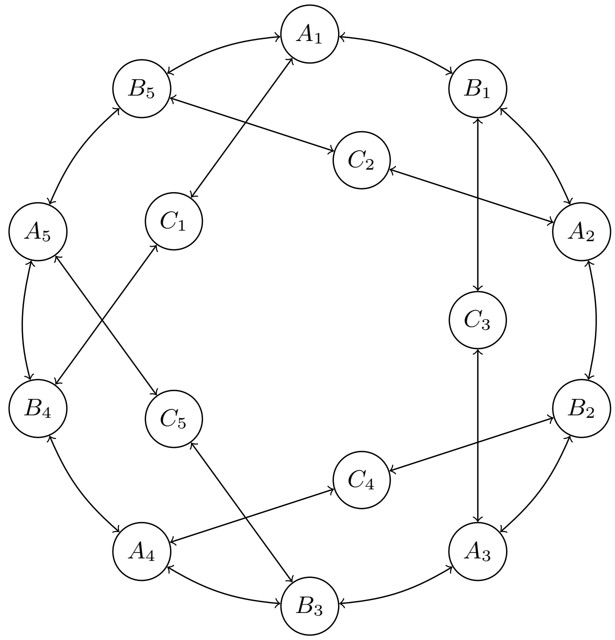

Figure 5: A simple symmetric FHG in which no core stable partition exists. For all $l \in \{1, \dots, 5\}$, A_l and C_l denote cliques of 3 agents and B_l denotes a clique of 2 agents. An edge from one clique to another denotes that every agent in the first clique is connected to every agent in the second clique. All depicted edges have weight 1. All missing edges have weight 0.

THEOREM 4. *It is coNP-complete to decide whether a given profile of valuation functions induces strict preferences over coalitions.*

PROOF. We show that it is NP-complete to decide whether, for an agent $i \in N$, there are two coalitions $S, T \in \mathcal{N}_i$ such that $S \sim_i T$. This implies the statement. First, note that this problem is clearly in NP, since, given two sets in $\mathcal{N}_i$, it can be checked in linear time whether i has the same valuation for both. To prove hardness, we provide a reduction from an instance $M = \{m_0, \dots, m_k\} \subseteq \mathbb{N}^{k+1}$ of *equal sum of subsets of equal cardinality (ESS)*. The answer to ESS is "Yes" if there are two distinct sets $S, T \subseteq \{1, \dots, k\}$ such that $|S| = |T|$ and $\sum_{i \in S} m_i = \sum_{i \in T} m_i$ and "No" otherwise. Cieliebak et al. [10] showed that ESS is NP-complete. Without loss of generality we assume $m_0 = 0$. Let $m_+ = \max_{s \in \{1,\dots,k\}} m_s, m_- = \min_{s \in \{1,\dots,k\}} m_s$, and $C = (k^2 + 2k)(m_+ - m_-) + 1$. We define $G = (N, N \times N, v)$ where $N = \{0, \dots, k\}$ and $v(\{i, j\}) = C + m_{i+j \pmod{k+1}}$ for all distinct $i, j \in \{0, \dots, k\}$.

Without loss of generality, we consider agent 0, who has valuation $C + m_i$ for all $i \in \{1, \dots, k\}$. Suppose there are two nonempty, distinct sets $S, T \subseteq M$ such that $|S| = |T|$, $0 \in S \cap T$, and $\sum_{i \in S} m_i = \sum_{i \in T} m_i$. Then, we have

$$v_0(S) = \frac{\sum_{i \in S}(C + m_i)}{|S| + 1} = \frac{|S|C + \sum_{i \in S} m_i}{|S| + 1} = \frac{|T|C + \sum_{i \in T} m_i}{|T| + 1} = \frac{\sum_{i \in T}(C + m_i)}{|T| + 1} = v_0(T).$$

Hence, we have $S \sim_0 T$.

For the other direction, we first state (without proof) that, for all $l \in \{2, \dots, n\}$,

$$\frac{l-1}{l}(C + m_-) > \frac{l-2}{l-1}(C + m_+). \quad (1)$$

Now suppose there exist two distinct coalitions $S, T \in \mathcal{N}_0$ such that $S \sim_0 T$. Assume that $|S| > |T|$. Then,

$$\begin{aligned} v_0(S) &= \frac{\sum_{i \in S}(C + m_i)}{|S|} \geq \frac{(|S| - 1)(C + m_-)}{|S|} \\ &> \frac{(|S| - 2)(C + m_+)}{|S| - 1} \geq \frac{(|T| - 1)(C + m_+)}{|T|} \\ &\geq \frac{\sum_{i \in T}(C + m_i)}{|T|} = v_0(T). \end{aligned}$$

The strict inequality follows from (1). This implies $S \succ_0 T$, contradiction our assumption, and hence $|S| = |T|$. Thus, $S \sim_0 T$ if and only if $|S| = |T|$ and $\sum_{i \in S} m_i = \sum_{i \in T} m_i$. $\square$

5.2 Existence of Stable Partitions

In the initial work on FHGs, Aziz et al. [3] showed hardness of deciding whether a given partition is core stable. For Nash stability and individual stability, this problem can easily be solved in polynomial time. In this section, we discuss problems of a similar spirit. We consider the problem of deciding whether a given FHG admits a stable partition for core stability, Nash stability, and individual stability. It turns out that this problem is hard whenever it is not trivial. This also implies that finding stable partitions is intractable.

First, we define the corresponding decision problems.

DEFINITION 1. *For a stability notion $\mathcal{E} \in \{CS, NS, IS\}$, the decision problem* (SYMM)FHG-$\mathcal{E}$ *is given by a (symmetric) FHG $(N, \succsim)$. The answer to (SYMM)FHG-$\mathcal{E}$ is "Yes" if there is an $\mathcal{E}$-stable partition in $(N, \succsim)$ and "No" otherwise.*

Sung and Dimitrov [20] proved that it is NP-hard to decide whether a given additively separable hedonic game admits a core stable, Nash stable, or individually stable partition. To this end, they provided a polynomial time reduction from the NP-complete problem *exact cover by 3-sets (E3C)* [cf. 15]. The construction can be adapted to obtain hardness results for FHGs. We explain the adaption to SYMMFHG-NS and FHG-IS.

An instance of E3C is a pair $(R, \mathcal{S})$ where R is a set such that $|R| = 3m$ for some positive integer m and $\mathcal{S}$ is a collection of subsets of R such that $|s| = 3$ for every $s \in \mathcal{S}$. The answer to E3C is "Yes" if there is a subset of $\mathcal{S}$ which partitions R, i.e., there is $\mathcal{S}' \subseteq \mathcal{S}$ such that $\bigcup_{s \in \mathcal{S}'} s = R$ and $s \cap s' = \emptyset$ for all distinct $s, s' \in \mathcal{S}'$. E3C remains NP-complete even if every $r \in R$ occurs in at most three elements of $\mathcal{S}$ [cf. 15]. Furthermore we can assume without loss of generality that every $r \in R$ occurs in at least one element of $\mathcal{S}$, otherwise the answer to the question is trivially "No".

Now, we construct for a given instance $(R, \mathcal{S})$ of E3C a weighted graph representing an FHG that admits a stable partition if and only the answer to E3C is "Yes". We start by constructing a subgraph G_s for every $s \in \mathcal{S}$. For every $s = \{u, v, w\}$, $G_s = (N_s, N_s \times N_s, v_s)$ where $N_s = \{\tau^s, \sigma_u^s, \sigma_v^s, \sigma_w^s\}$ and $v_s(i, j) = 1$ for all $i, j \in N_s$. Figure 6 illustrates such a subgraph.

Every subset $\mathcal{S}' \subseteq \mathcal{S}$ can be identified with the set of graphs $\{G_s\}_{s \in \mathcal{S}'}$. Next we compute, for every $r \in R$, the number

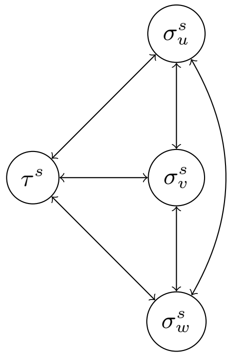

Figure 6: The subgraph corresponding to $s = \{u, v, w\} \in \mathcal{S}$. All edges have weight 1.

$l_r = |\{s \in \mathcal{S} : r \in \mathcal{S}\}| - 1$, i.e., the number of 3 sets containing r after removing an R-partitioning subset $\mathcal{S}'$ form $\mathcal{S}$. Since we assumed that every $r \in R$ is contained in at least one $s \in \mathcal{S}$, we have $l_r \geq 0$ for every $r \in R$. For every $r \in R$ we add l_r graphs $G_{r,k}$, $k \in \{1, \dots, l_r\}$. The exact structure of the $G_{r,k}$'s depends on the actual proof, but in general they have to fulfill the following conditions.

- The FHG induced by $G_{r,k}$ does not admit a stable partition
- The set of vertices of $G_{r,k}$ contains a vertex α_r^k such that the FHG induced by $G_{r,k}$ admits a stable partition when α_r^k is removed from the game.

The last step is to connect every σ_r^s to every α_r^k by an edge of weight 1 (and *vice versa*) for every $r \in R$. All vertices which are not connected by an edge of weight one are connected by an edge of weight $-M$, where M is larger than the sum of the weights of adjacent edges with positive weight for every vertex. Notice that M does not depend on the given instance of E3C. For our purposes, $M = 20$ suffices. The whole graph now induces an FHG. Notice that this game is symmetric if all $G_{r,k}$'s are symmetric. Figure 7 illustrates the construction for a small instance of E3C. The $G_{r,k}$'s are obtained from the graph depicted in Figure 2, which is an example of an FHG that does not admit an individually stable partition.

The main idea behind the whole construction is the following. For every $r \in R$, l_r of the σ_r^s's are needed to "stabilize" the $G_{r,k}$'s. On the other hand, the G_s's admit a stable partition only if the whole subgraph forms a coalition or if the σ_r^s's are in a coalition with a corresponding α_r^k. In a stable partition, for every $r \in R$, exactly one σ_r^s is not in a coalition with some α_r^k, but instead in a coalition consisting of every vertex in G_s. All these G_s's together can be identified with a subset $\mathcal{S}' \subseteq \mathcal{S}$ which is a partition of R. A stable partition exists if and only if such an $\mathcal{S}'$ exists.

Our findings on the computational complexity of checking whether a stable partition exists are summarized in the following theorem.

THEOREM 5. *The following hardness results hold:*

(i) SYMMFHG-CS is NP-hard,

(ii) SYMMFHG-NS is NP-complete, and

(iii) FHG-IS is NP-complete.

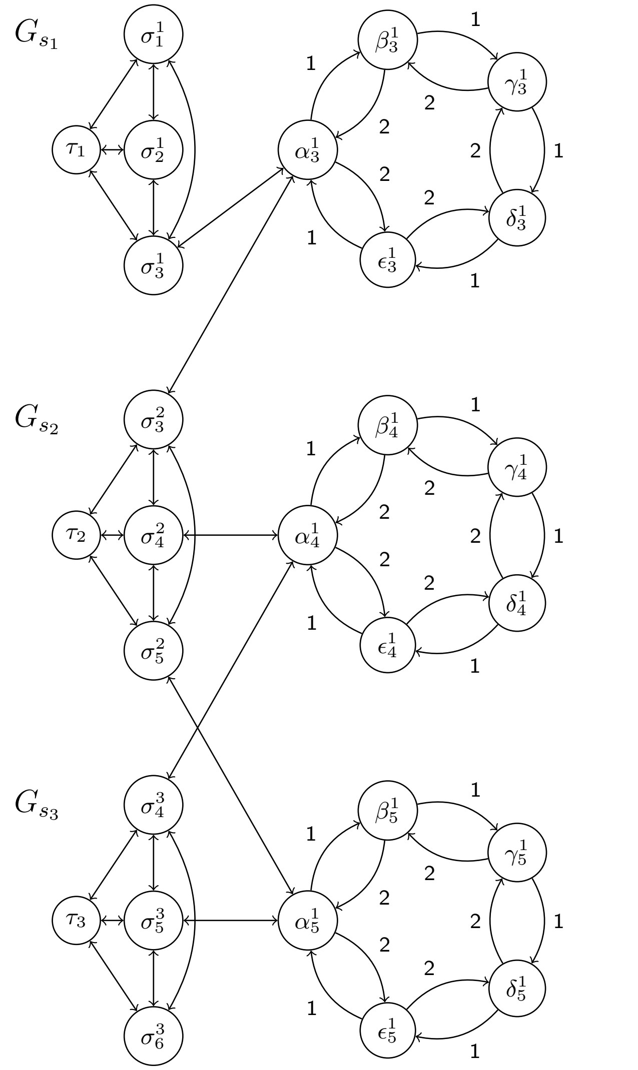

Figure 7: The graph belonging to the reduction of an instance $(R, \mathcal{S})$ of E3C, where $R = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{S} = \{\{1, 2, 3\}, \{3, 4, 5\}, \{4, 5, 6\}\}$. All unlabeled edges have weight one. All edges that are not depicted have weight -20.

PROOF. Due to space constraints we only give a proof for (iii) here. First note that it is easy to verify if a partition is individually stable. For every agent, one can check in polynomial time if he can deviate without making an agent in his new coalition worse off. So FHG-IS is in NP.

Now we will go through the reduction described above. Let $(R, \mathcal{S})$ be an instance of E3C. An instance of FHG-IS is constructed as follows. Let $N = \{\sigma_r^s : s \in \mathcal{S}, r \in s\} \cup \{\tau^s : s \in \mathcal{S}\} \cup \{\alpha_r^k, \beta_r^k, \gamma_r^k, \delta_r^k, \epsilon_r^k : r \in R, 1 \leq k \leq l_r\}$. The σ_r^s's and τ^s's form the G_s's and the α_r^k's to ϵ_r^k's the $G_{r,k}$'s, respectively. The agents valuation functions are defined as follows.

(i) For all $s \in \mathcal{S}, r \in s$: $v_{\sigma_r^s}(\tau^s) = v_{\tau^s}(\sigma_r^s) = 1$

(ii) For all $s \in \mathcal{S}, r, r' \in s, r \neq r'$: $v_{\sigma_r^s}(\sigma_{r'}^s) = v_{\sigma_{r'}^s}(\sigma_r^s) = 1$

(iii) For all $r \in R, s \in \mathcal{S}, r \in s, k \in \{1, \dots, l_r\}$: $v_{\sigma_r^s}(\alpha_r^k) = v_{\alpha_r^k}(\sigma_r^s) = 1$

(iv) For all $r \in R, k \in \{1, \ldots, l_r\}$: $v_{\alpha_r^k}(\beta_r^k) = v_{\beta_r^k}(\gamma_r^k) = v_{\gamma_r^k}(\delta_r^k) = v_{\delta_r^k}(\epsilon_r^k) = v_{\epsilon_r^k}(\alpha_r^k) = 1$ and $v_{\alpha_r^k}(\epsilon_r^k) = v_{\epsilon_r^k}(\delta_r^k) = v_{\delta_r^k}(\gamma_r^k) = v_{\gamma_r^k}(\beta_r^k) = v_{\beta_r^k}(\alpha_r^k) = 2$.

(v) For all remaining pairs (i, j) we define $v_i(j) = -20$.

The number of agents is $3|\mathcal{S}| + |\mathcal{S}| + 5(3|\mathcal{S}| - |R|)$ and all valuation functions are bounded by a constant, so this construction can be computed in polynomial time.

First suppose there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ which is a partition of R. For every $r \in R$, let $\{s_r^1, s_r^2, \ldots, s_r^{l_r}\} = \{s \in \mathcal{S} \setminus \mathcal{S}' : r \in s\}$ be an enumeration of the sets outside of $\mathcal{S}'$ containing r. Now consider the following partition.

$$\pi = \{\{\tau^s\} \cup \{\sigma_r^s : r \in s\} : s \in \mathcal{S}'\} \cup \{\{\tau^s\} : s \in \mathcal{S} \setminus \mathcal{S}'\}$$
$$\cup \{\{\sigma_r^{s_r^k}, \alpha_r^k\}, \{\beta_r^k, \gamma_r^k\}\{\delta_r^k, \epsilon_r^k\} : r \in R, k \in \{1, \ldots, l_r\}\}$$

We claim that π is individually stable.

- Consider an agent $i \in \{\sigma_r^s : r \in R, s \in \mathcal{S}\}$.
  - If $S_\pi(i) = \{\tau^s\} \cup \{\sigma_r^s : r \in s\}$ then $v_i(\pi(i)) = 3/4$ and for all $S_k \in \pi \setminus \{\pi(i)\}$ we have $v_i(S_k \cup \{i\}) < 3/4$, and
  - if $S_\pi(i) = \{\sigma_r^{s_r^k}, \alpha_r^k\}$ then $v_i(\pi(i)) = 1/2$ and for all $S_k \in \pi \setminus \{\pi(i)\}$ we have $v_i(S_k \cup \{i\}) \leq 1/2$.

  So i has no incentive to deviate.
- Consider an agent $i \in \{\tau^s : s \in \mathcal{S}\}$. Then $v_i(\pi(i)) \geq 0$ and for all $S_k \in \pi \setminus \{\pi(i)\}$ we have $v_i(S_k \cup \{i\}) < 0$. So i has no incentive to deviate.
- Consider an agent $i \in \{\alpha_r^k, \beta_r^k, \gamma_r^k, \delta_r^k, \epsilon_r^k : r \in R, k \in \{1, \ldots, l_r\}\}$. We have $v_i(\pi(i)) > 0$ and for all $S_k \in \pi \setminus \{\pi(i)\}$ we have $v_i(S_k \cup \{i\}) < 0$. So i has no incentive to deviate.

Hence, π is Nash stable and thus individually stable.

For the other direction, suppose there exists an individually stable partition π. For every $r \in R$ and $k \in \{1, \ldots, l_r\}$, $G_{r,k}$ is isomorphic to the graph depicted in Figure 2. So in the FHG we constructed from the E3C instance, a partition can only be stable, if, for every $r \in R$ and every $k \in \{1, \ldots, l_r\}$, there exists an agent $i \in \{\alpha_r^k, \beta_r^k, \gamma_r^k, \delta_r^k, \epsilon_r^k\}$ such that $\pi(i) \not\subseteq \{\alpha_r^k, \beta_r^k, \gamma_r^k, \delta_r^k, \epsilon_r^k\}$. By (ii) and (iii) in the definition of the valuation functions, the only candidate for this is α_r^k. From this we directly get

$$\{\{\beta_r^k, \gamma_r^k\}\{\delta_r^k, \epsilon_r^k\} : r \in R, k \in \{1, \ldots, l_r\}\} \subseteq \pi$$

and

$$\pi(\alpha_r^k) \subseteq \{\alpha_r^k\} \cup \{\sigma_r^s : s \in \mathcal{S}\} \text{ for all } r \in R, k \in \{1, \ldots, l_r\}.$$

Suppose $|\pi(\alpha_r^k)| > 2$, then there exist distinct $s, s' \in \mathcal{S}$ such that $\{\sigma_r^s, \sigma_r^{s'}\} \subseteq \pi(\alpha_r^k)$. But then π cannot be individually stable since σ_r^s would rather be alone than a member of $\pi(\alpha_r^k)$. Hence, for every $r \in R$ and $k \in \{1, \ldots l_r\}$, there is an agent σ_r^s such that $\pi(\alpha_r^k) = \{\alpha_r^k, \sigma_r^s\}$.

By (v) in the definition of the valuation functions, we get $\pi(\tau^s) \subseteq \{\tau^s\} \cup \{\sigma_r^s : r \in s\}$. We can conclude $\pi(\tau^s) = \{\tau^s\}$ or $\pi(\tau^s) = \{\tau^s\} \cup \{\sigma_r^s : r \in s\}$, because otherwise there exists $r \in R$ such that $v_{\sigma_r^s}(\pi(\sigma_r^s)) \leq 1/2 < 2/3 \leq v_{\sigma_r^s}(\pi(\tau^s) \cup \{\sigma_r^s\})$ and $v_i(\pi(\tau^s) \cup \{\sigma_r^s\}) > v_i(\pi(\tau^s))$ for every $i \in \pi(\tau^s)$. Furthermore, we have, for every $i \in \{\sigma_r^s : s \in \mathcal{S}, r \in s\}$, that $\pi(i) \neq \{i\}$, otherwise $v_i(\pi(i)) = 0 < 1/2 \leq v_i(\pi(\tau^s) \cup \{i\})$ and $v_j(\{\tau^s, i\}) > v_j(\{\tau^s\})$ for all $j \in \pi(\tau^s)$.

Exactly l_r agents in $\{\sigma_r^s : s \in \mathcal{S}\}$ form a coalition with some α_r^k for every $r \in R$. It follows that, for every $r \in R$, there is exactly one $s \in \mathcal{S}$ such that $\pi(\sigma_r^s) = \{\tau^s\} \cup \{\sigma_{r'}^s : r' \in s\}$. Hence, $\mathcal{S}' = \{s \in \mathcal{S} : \pi(\tau^s) = \{\tau^s\} \cup \{\sigma_r^s : r \in s\}\}$ is a partition of R.

The construction above does not work for core stability since the partition given by a solution to the corresponding instance of E3C would be blocked by the subgraphs G_s. However, Sung and Dimitrov [20] provided a slightly different construction for core stability in additively separable hedonic games which can be adapted to symmetric FHGs. The FHG depicted in Figure 3 serves as gadget for this construction. □

If there exists a symmetric FHG that does not admit an individually stable partition, the construction from Theorem 5 can be used to prove NP-completeness of SYMMFHG-IS. Every smallest symmetric FHG that does not admit an individually stable partition could serve as a gadget. If, on the other hand, no such game exists, the answer to SYMMFHG-IND is trivially "Yes" for every game.

6. CONCLUSIONS

We studied core stability and stability notions based on deviations of a single agent, i.e., Nash stability and individual stability. For these stability notions we examined whether a stable partition may fail to exist for three classes of FHGs. In particular, we showed that core stable partitions may not exist in simple symmetric FHGs. This answers a question proposed by Aziz et al. [3].

In the second part of the paper, we leveraged the nonexistence examples to show that deciding the existence (and thus also finding) stable partitions for the corresponding notion of stability and class of FHGs is NP-hard. By contrast, Aziz et al. [3] proposed a number of classes of FHGs in which stable partitions are guaranteed to exist by providing polynomial-time algorithms for computing such partitions. These results suggest a strong connection between the existence of stable partitions and the hardness of finding stable partitions. It is an interesting problem whether this connection can be made more precise and extended to more general classes of hedonic games.

Since our results show that for large classes of FHGs the existence of a stable partition cannot be guaranteed, it would be desirable to find more natural classes for which the existence of stable partitions is guaranteed. In particular, the existence of individually stable partitions in symmetric FHGs remains an open problem.

Acknowledgments

We are grateful to Haris Aziz, Paul Harrenstein, and the anonymous reviewers for helpful comments. This material is based upon work supported by Deutsche Forschungsgemeinschaft under grants BR 2312/7-2 and BR 2312/10-1.

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