Casting the lot puts an end to disputes and decides between powerful contenders.

— Solomon, Proverbs 18:18

Fishburn's Maximal Lotteries

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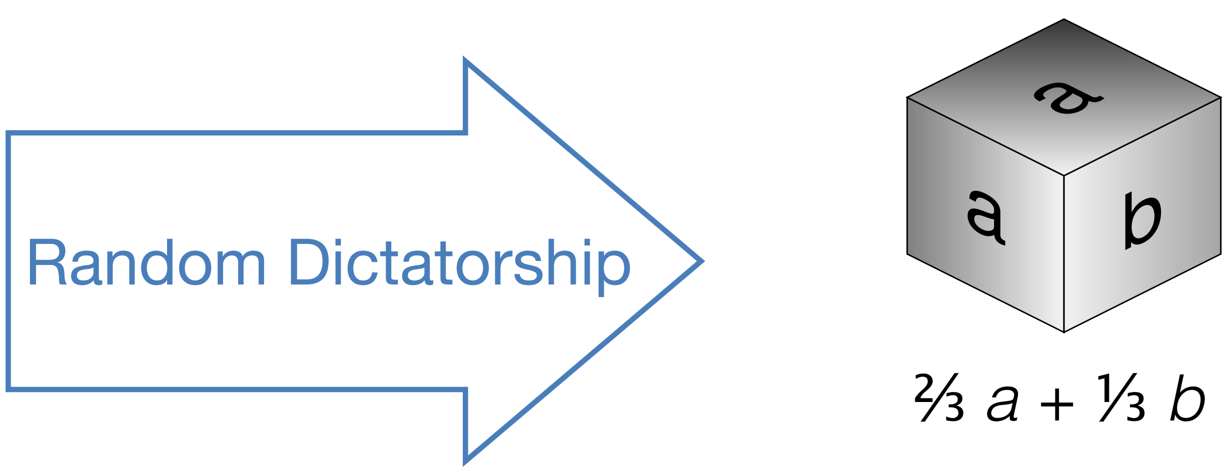
DSS
Decision Sciences & Systems

Probabilistic Social Choice

- Voters have complete and transitive preference relations $\succcurlyeq_i$ over a finite set of alternatives A.
- A *social decision scheme* f maps a preference profile $(\succcurlyeq_1, \dots, \succcurlyeq_n)$ to a lottery $\Delta(A)$.
  - randomization or other means of tie-breaking are inevitable when insisting on anonymity and neutrality.
  - first studied by Zeckhauser (1969), Fishburn (1972), Intriligator (1973), Nitzan (1975), and Gibbard (1977)

1	**1**	**1**
a	*b*	*a*
b	*a*	*c*
c	*c*	*b*



⅔ a + ⅓ b

Germain Kreweras

Maximal Lotteries

Peter C. Fishburn

- Kreweras (1965) and Fishburn (1984)
  - rediscovered by Laffond et al. (1993), Felsenthal and Machover (1992), Fisher and Ryan (1995), Rivest and Shen (2010)
- Let $(M_{x,y})$ be the majority margin matrix, i.e., $M_{x,y} = |\{i : x \succcurlyeq_i y\}| - |\{i : y \succcurlyeq_i x\}|$.
- M admits a (weak) Condorcet winner if M contains a non-negative row, i.e., there is a standard unit vector v such that $v^T M \geq 0$.

1	1	1
a	b	c
b	a	a
c	c	b

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \end{array} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \geq 0$$

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1	1	1
a	b	c
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$$\begin{pmatrix}1\\0\\0\end{pmatrix}^T \begin{array}{c} \\ a\\ b\\ c\end{array}\!\!\begin{array}{c}\begin{array}{ccc} a & b & c\end{array}\\ \begin{pmatrix}0 & 1 & 1\\ -1 & 0 & 1\\ -1 & -1 & 0\end{pmatrix}\end{array} = \begin{pmatrix}0 & 1 & 1\end{pmatrix} \geq 0$$

- A lottery p is maximal if $p^T M \geq 0$.
  - randomized Condorcet winner
  - p is “at least as good” as any other lottery
  - bilinear expected majority margin $p^T M\, q \geq 0$ for all $q \in \Delta(A)$

1	1	1
a	b	c
b	c	a
c	a	b

$$\begin{pmatrix}1/3\\1/3\\1/3\end{pmatrix}^T \begin{array}{c} \\ a\\ b\\ c\end{array}\!\!\begin{array}{c}\begin{array}{ccc} a & b & c\end{array}\\ \begin{pmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{pmatrix}\end{array} = \begin{pmatrix}0 & 0 & 0\end{pmatrix} \geq 0$$

Germain Kreweras

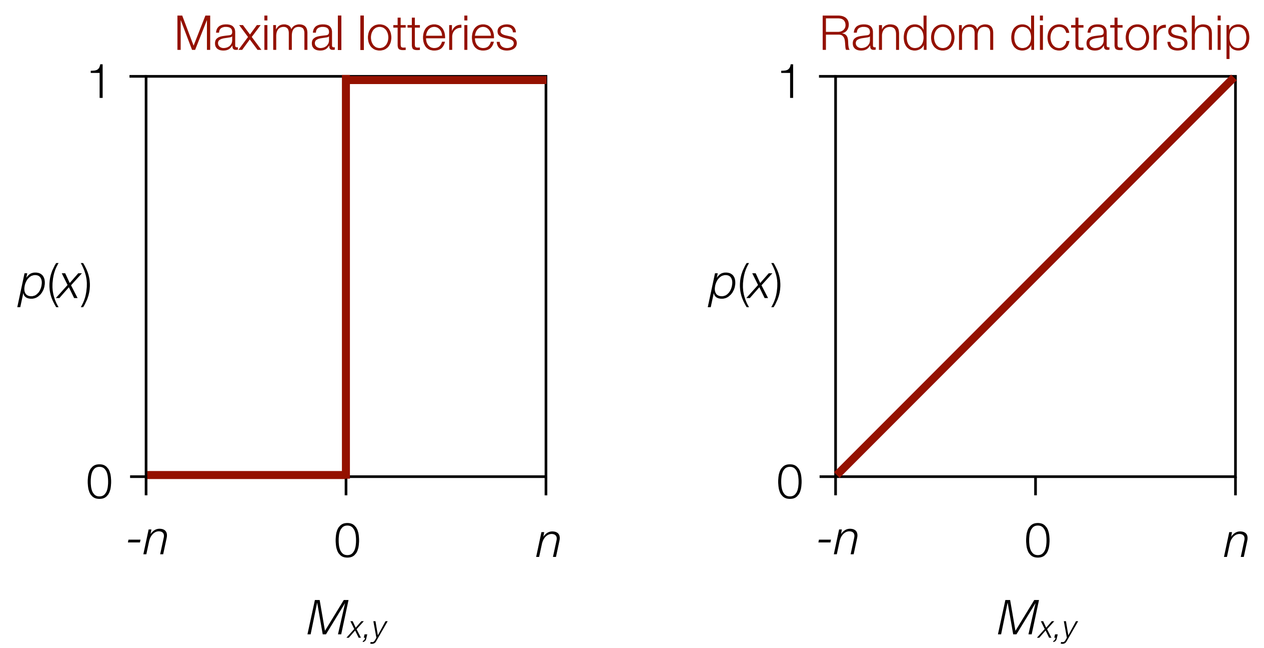
Maximal Lotteries

Peter C. Fishburn

- always exist due to Minimax Theorem (v. Neumann, 1928)
- almost always unique
  - set of profiles with multiple maximal lotteries has measure zero
  - always unique for odd number of voters with strict preferences (Laffond et al., 1997)
- do not require asymmetry, completeness, or even transitivity of individual preferences
- can be efficiently computed via linear programming
- known as popular mixed matchings in assignment (aka house allocation) domain (Kavitha et al., 2011)

Examples

- Two alternatives



- *M* can be interpreted as a symmetric zero-sum game.
  - Maximal lotteries are mixed minimax strategies.

2	2	1
a	*b*	*c*
b	*c*	*a*
c	*a*	*b*

	a	*b*	*c*
a	0	1	-1
b	-1	0	3
c	1	-3	0

  - The unique maximal lottery is $\frac{3}{5}\, a + \frac{1}{5}\, b + \frac{1}{5}\, c$.

	Maximal Lotteries	Random Serial Dictatorship	Borda's Rule
population-consistency			
agenda-consistency			
cloning-consistency			
Condorcet-consistency			
(SD-) **strategyproofness**			
(ST-) **group-strategyproofness**			
(SD-) **participation**			
(SD-) **efficiency**			
efficient computability			
randomness			

	Maximal Lotteries	**Random Serial Dictatorship**	**Borda's Rule**
population-consistency	✅	only for strict prefs	✅
agenda-consistency	✅	✅	—
cloning-consistency	✅ even composition-consistency	✅	—
Condorcet-consistency	✅	—	—
(SD-) **strategyproofness**	—	✅ even strongly	—
(ST-) **group-strategyproofness**	✅	✅	—
(SD-) **participation**	✅ even PC-group-participation	✅ even very strongly	✅
(SD-) **efficiency**	✅	only for strict prefs otherwise only *ex post*	✅
efficient computability	✅	#P-complete in P for strict prefs	✅
randomness	*some*	*a lot*	*very little*

Population-Consistency ✅

Whenever two disjoint electorates agree on a lottery,
this lottery should also be chosen by the union of both electorates.

1	1
a	*b*
b	*c*
c	*a*

R

½ *a* + ½ *b*

1	1
a	*b*
c	*c*
b	*a*

S

½ *a* + ½ *b*

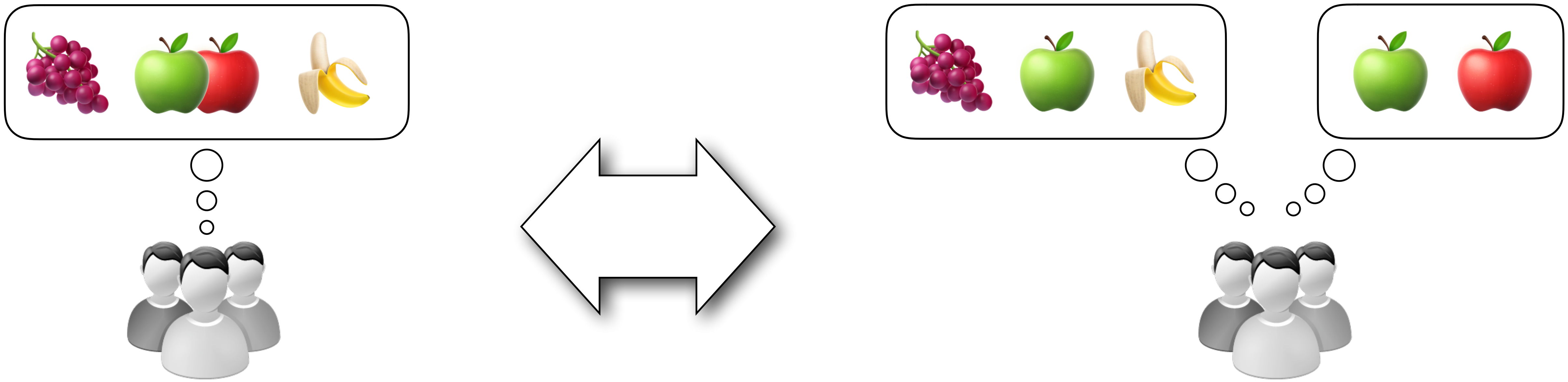
1	1	2
a	*a*	*b*
b	*c*	*c*
c	*b*	*a*

$R \cup S$

½ *a* + ½ *b*

- first proposed by Smith (1973), Young (1974), Fine & Fine (1974)
- also known as “reinforcement” (Moulin, 1988)
- famously used for the characterization of scoring rules and Kemeny

Composition-Consistency ✅



Composition-Consistency ✅

Decomposable preference profiles are treated component-wise.

In particular, alternatives are not affected by the cloning of other alternatives

2	1	3
a	a	b
b'	b	b'
b	b'	a

R

$\frac{1}{2}a + \frac{1}{3}b + \frac{1}{6}b'$

3	3
a	b
b	a

$R|_A$

$\frac{1}{2}a + \frac{1}{2}b$

2	4
b'	b
b	b'

$R|_B$

$\frac{2}{3}b + \frac{1}{3}b'$

$A=\{a,b\}$

$B=\{b,b'\}$

- Laffond, Laslier, and Le Breton (1996)
- cloning consistency precursors: Arrow and Hurwicz (1972), Maskin (1979), Moulin (1986), Tideman (1987)


Chevalier de Borda

Non-Probabilistic Social Choice


Marquis de Condorcet

- All scoring rules satisfy population-consistency.
(Smith 1973; Young, 1974)
- No Condorcet extension satisfies population-consistency.
(Young and Levenglick, 1978)
- Many Condorcet extensions satisfy composition-consistency. (Laffond et al., 1996)
- No Pareto-optimal scoring rule satisfies composition-consistency. (Laslier, 1996)
- Population-consistency and composition-consistency are incompatible in non-probabilistic social choice. (Brandl et al., 2016)
- A probabilistic SCF satisfies population-consistency and composition-consistency iff it returns all maximal lotteries.
(Brandl et al., 2016)

Agenda Consistency ✅

> A lottery should be chosen from two agendas iff it is also chosen in the union of both agendas.

1	1
a	*b*
d	*c*
b	*d*
c	*a*

R

$\frac{1}{2}a + \frac{1}{2}b$

1	1
a	*b*
b	*c*
c	*a*

$R|_A$

$\frac{1}{2}a + \frac{1}{2}b$

1	1
a	*b*
d	*d*
b	*a*

$R|_B$

$\frac{1}{2}a + \frac{1}{2}b$

$A=\{a,b,c\}$

$B=\{a,b,d\}$

- Sen (1971)'s α (contraction) and γ (expansion)
- at the heart of numerous impossibilities (e.g., Blair et al., 1976; Sen, 1977; Kelly, 1978; Schwartz, 1986)

SD-Participation ✅

No agent can obtain more expected utility (for all vNM representations) by abstaining from an election.

	a	b	c
a	0	1	-1
b	-1	0	1
c	1	-1	0

1	**1**	**2**	**1**
a	*a*	*b*	*c*
c	*b*	*c*	*a*
b	*c*	*a*	*b*

R

⅓ *a* + ⅓ *b* + ⅓ *c*

1	**2**	**1**
a	*b*	*c*
b	*c*	*a*
c	*a*	*b*

R'

b

	a	b	c
a	0	0	-2
b	0	0	2
c	2	-2	0

- cannot be satisfied by *resolute* Condorcet extensions (Moulin, 1988)
- satisfied by maximal lotteries

SD-Efficiency ✅

The expected utility of a voter can only be increased by decreasing the expected utility of another.

- maximal lotteries are SD-efficient
- violated by random serial dictatorship: there can even be lotteries that give strictly more expected utility to *all* voters!
- maximal lotteries are social-welfare-maximizing lotteries for canonical skew-symmetric bilinear (SSB) utility functions

SD-Strategyproofness

No agent can obtain more expected utility (for all vNM representations) by misreporting his preferences.

	a	b	c
a	0	1	-1
b	-1	0	1
c	1	-1	0

1	1	2	1
a	a	b	c
c	b	c	a
b	c	a	b

R

1	1	2	1
a	a	b	c
b	b	c	a
c	c	a	b

R'

	a	b	c
a	0	1	-1
b	-1	0	3
c	1	-3	0

$p = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c$

$q = \frac{3}{5}a + \frac{1}{5}b + \frac{1}{5}c$

- maximal lotteries are *not* strategyproof with respect to stochastic dominance
  - *q* will always yield more expected utility than *p*

SD-Strategyproofness (ctd.)

- Maximal lotteries are SD-strategyproof in all profiles that admit a Condorcet winner (Peyre, 2013) ✅.
- Maximal lotteries are group-strategyproof with respect to the “sure thing” lottery extension ✅.
  - loosely based on Savage’s sure-thing principle
  - ignore alternatives that receive the same probability in p and q
  - all remaining alternatives in the support of q should be preferred to all remaining alternatives in the support of p.
- Almost all randomized versions of classic rules fail to satisfy even this weak notion of strategyproofness
  - e.g., Borda, Copeland, STV, Kemeny, Dodgson

References

pnyx.dss.in.tum.de

- Aziz, Brandl, and B. *Universal dominance and welfare for plausible utility functions*. **Journal of Mathematical Economics**, 60:123-133, 2015.
- Aziz, B., and Brill. *The computational complexity of random serial dictatorship*. **Economics Letters**, 121(3):341-345, 2013.
- Aziz, B., and Brill. *On the tradeoff between efficiency and strategyproofness*. 2016. Working paper.
- Brandl, B., and Hofbauer. *Welfare maximization entices participation*. 2016. Working paper.
- Brandl, B., and Suksompong. *The impossibility of extending random dictatorship to weak preferences*. **Economics Letters**, 141:44-47, 2016.
- Brandl, B., and Seedig. *Consistent probabilistic social choice*. **Econometrica**, 84(5):1839-1880, 2016.
- Fishburn. *Probabilistic social choice based on simple voting comparisons.* **Review of Economic Studies**, 51(4):683-692, 1984.